CONNECTEDNESS OF THE BALMER SPECTRUM OF THE RIGHT BOUNDED DERIVED CATEGORY OF A COMMUTATIVE NOETHERIAN RING

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Abstract. By virtue of Balmer's celebrated theorem, the classification of thick tensor ideals of a tensor triangulated category $\mathcal{T}$ is equivalent to the topological structure of its Balmer spectrum $\mathsf{Spc}\,\mathcal{T}$. Motivated by this theorem, we discuss connectedness and irreducibility of the Balmer spectrum of a right bounded derived category of finitely generated modules over a commutative ring.

1. Introduction

The content of this article is based on the paper [7].

Tensor triangulated geometry is a theory introduced by Balmer [1] to study tensor triangulated categories by algebro-geometric methods. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be an essentially small tensor triangulated category (i.e., a triangulated category $\mathcal{T}$ equipped with a symmetric monoidal tensor product $\otimes$ which is compatible with the triangulated structure). Then Balmer defined a topological space $\mathsf{Spc}\,\mathcal{T}$ which we call the Balmer spectrum of $\mathcal{T}$. A celebrated theorem due to Balmer [1] states that the radical thick tensor ideals of $\mathcal{T}$ are classified using the geometry of $\mathsf{Spc}\,\mathcal{T}$:

Theorem 1 (Balmer)**.** *There is an order-preserving one-to-one correspondence*

$$\{\textit{radical thick tensor ideals of } \mathcal{T}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{Thomason subsets of } \mathsf{Spc}\mathcal{T}\},$$

where f and g are given by $f(\mathcal{X}) := \mathsf{Spp}\,\mathcal{X} := \bigcup_{X\in\mathcal{X}} \mathsf{Spp}\,X$ and $g(W) := \mathsf{Spp}^{-1}(W) := \{X \in \mathcal{T} \mid \mathsf{Spp}X \subseteq W\}$, respectively.

From this result, if we want to classify the radical thick tensor ideals of a given tensor triangulated category $\mathcal{T}$, we have only to understand the topological space $\mathsf{Spc}\,\mathcal{T}$. Therefore, it is crucial to discuss topological properties of the Balmer spectrum.

In this article, we consider the right bounded derived category $\mathsf{D}^-(\mathsf{mod}R)$ of a commutative Noetherian ring R. This triangulated category is a tensor triangulated category with respect to derived tensor product, and we can consider its Balmer spectrum $\mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R)$.

The detailed version [7] of this paper has been published in Journal of Pure and Applied Algebra.

2. Tensor Triangular Geometry

In this section, let us recalling Balmer's tensor triangular geometry [1]. Throughout this section, fix a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$. The following categories are examples of tensor triangulated categories which appear in algebra.

Example 2. (1) Let X be a scheme. Then the derived category $\mathsf{D}^{\mathsf{perf}}(X)$ of perfect complexes of X forms a tensor triangulated category together with derived tensor product $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$ and unit $\mathcal{O}_X$.

(2) Let k be a field and G a finite group. Then the stable category $\underline{\mathsf{mod}}\, R$ forms a tensor triangulated category together with tensor product $\otimes_k$ and unit k.

(3) Let R be a commutative noetherian ring. Then the right bounded derived category $\mathsf{D}^-(\mathsf{mod}\, R)$ is a tensor triangulated category with respect to derived tensor product $\otimes^{\mathbb{L}}_R$ and unit R.

One can define the notions of *thick tensor ideals*, *radical thick tensor ideals* and *prime thick tensor ideals*, which behave similarly to ideals, radical ideals, and prime ideals of a commutative ring.

Definition 3. (cf. [1, Definitions 1.2, 2.1 and 4.1])

(1) We say that an addtive full subcategory $\mathcal{X}$ of $\mathcal{T}$ is a *thick tensor ideal* if
   (a) $\mathcal{X}$ is triangulated: for any triangle $L \to M \to N \to L[1]$ in $\mathcal{T}$, if two out of L, M, N belongs to $\mathcal{X}$, then so does the third,
   (b) $\mathcal{X}$ is thick: if $L \otimes M$ belongs to $\mathcal{X}$, then so do L and M,
   (c) $\mathcal{X}$ is an ideal: for any $L \in \mathcal{X}$ and $M \in \mathcal{T}$, their tensor product $L \otimes M \in \mathcal{T}$.

(2) For a thick tensor ideal $\mathcal{X}$, we denote by $\sqrt{\mathcal{X}}$ the *radical* of $\mathcal{X}$, that is, the subcategory of $\mathcal{T}$ consisting of objects M such that the n-fold tensor product $M^{\otimes n}$ belongs to $\mathcal{X}$ for some integer n.
   We say that a thick tensor ideal $\mathcal{X}$ is *radical* if $\sqrt{\mathcal{X}} = \mathcal{X}$.

(3) We say that a proper thick tensor ideal $\mathcal{P}$ is prime if $M \otimes N \in \mathcal{P}$ implies either $M \in \mathcal{P}$ or $N \in \mathcal{P}$. Denote by $\mathsf{Spc}\, \mathcal{T}$ the set of prime thick tensor ideals of $\mathcal{T}$.

Let us give one typical example of thick tensor ideals and prime thick tensor ideals for $\mathsf{D}^-(\mathsf{mod} R)$.

Example 4. For an object $M \in \mathsf{D}^-(\mathsf{mod} R)$, define its *cohomological support* by

$$\mathsf{Supp}\, M := \{\mathfrak{p} \in \mathsf{Spec}\, R \mid M_{\mathfrak{p}} \ncong 0 \text{ in } \mathsf{D}^-(\mathsf{mod} R_{\mathfrak{p}})\}$$

Then for any subset W of $\mathsf{Spec}\, R$,

$$\mathsf{Supp}^{-1}(W) := \{M \in \mathsf{D}^-(\mathsf{mod} R) \mid \mathsf{Supp}\, M \subseteq W\}.$$

is a prime thick tensor ideal. Indeed, it is easily verify that $\mathsf{Supp}^{-1}(W)$ is a thick tensor ideal since localization defines an exact functor.

Taking $W := \{\mathfrak{q} \in \mathsf{Spec}\, R \mod \mathfrak{q} \nsubseteq \mathfrak{p}\}$ for a fixed $\mathfrak{p}$, we have a prime thick tensor ideal

$$\mathcal{S}(\mathfrak{p}) := \{M \in \mathsf{D}^-(\mathsf{mod} R) \mid M_{\mathfrak{p}} \cong 0\}.$$

From the above argument, $\mathcal{S}(\mathfrak{p})$ is a thick tensor ideal. Moreover, it is prime by Nakayama's lemma.

Next, let us introduce a topology on $\mathsf{Spc}\,\mathcal{T}$, which is inspired the Zariski topology.

Definition 5. (cf. [1, Definition 2.1]) For a subset $\mathcal{E}$ of $\mathcal{T}$, define

$$\mathsf{Z}(\mathcal{E}) := \{\mathcal{P} \in \mathsf{Spc}\,\mathcal{T} \mid \mathcal{P} \cap \mathcal{E} = \emptyset\} \subseteq \mathsf{Spc}\,\mathcal{T}$$

Then one can easily check that the family $\{\mathsf{Z}(\mathcal{E}) \mid \mathcal{E} \subseteq \mathcal{T}\}$ forms a closed subsets of $\mathsf{Spc}\,\mathcal{T}$. Precisely, these subsets of $\mathsf{Spc}\,\mathcal{T}$ satisfies:

(1) $\mathsf{Z}(\emptyset) = \mathsf{Spc}\,\mathcal{T}, \;\; \mathsf{Z}(\mathcal{T}) = \emptyset$.
(2) $\mathsf{Z}(\mathcal{E}) \cup \mathsf{Z}(\mathcal{E}') = \mathsf{Z}(\mathcal{E} \oplus \mathcal{E}')$, where $\mathcal{E} \oplus \mathcal{E} = \{M \oplus M' \mid M \in \mathcal{E}, M' \in \mathcal{E}'\}$
(3) $\bigcap_{i\in I} \mathsf{Z}(\mathcal{E}_i) = \mathsf{Z}(\bigcup_{i\in I} \mathcal{E}_i)$.

Thus, $\mathsf{Spc}\,\mathcal{T}$ is a topological space with closed subsets $\mathsf{Z}(\mathcal{E})$. We call this topological space the *Balmer spectrum* of $\mathcal{T}$.

For an object $M \in \mathsf{T}$, define its *Balmer support* by

$$\mathsf{BSupp}\, M := \mathsf{Z}(\{M\}) = \{\mathcal{P} \in \mathsf{Spc}\,\mathcal{T} \mid M \notin \mathcal{P}\}.$$

It follows from the definition that $\mathsf{Z}(\mathcal{E}) = \bigcap_{M\in\mathcal{E}} \mathsf{BSupp}\, M$. This means that the Balmer supports $\mathsf{BSupp}\, M$ of objects forms a closed basis of the Balmer spectrum. For a subset $\mathcal{X}$ of $\mathcal{T}$, define

$$\mathsf{BSupp}\, \mathcal{X} = \bigcup_{M\in\mathcal{X}} \mathsf{BSupp}\, M,$$

which is a specialization closed subset of $\mathsf{Spc}\,\mathcal{T}$.

Balmer proved the following celebrated result which gives a classification of radical thick tensor ideals via Thomason subsets of the Balmer spectrum.

Theorem 6. [1, Theorem 4.10] *There is a bijection between*

(1) *the set of radical thick tensor ideals of $\mathcal{T}$, and*
(2) *the set of Thomason subsets (i.e., unions of complements of quasi-compact open subsets) of $\mathsf{Spc}\,\mathcal{T}$.*

Thus, the classification of radical thick tensor ideals is interpreted as the study of the topological space $\mathsf{Spc}\,\mathcal{T}$. Hence determining the topological structure is an important problem. Balmer also determined the Balmer spectra of tensor triangulated categories $\mathsf{D}^{\mathsf{perf}}(R)$, $\underline{\mathsf{mod}}\, kG$:

Example 7. [1, Corollaries 5.6 and 5.10]

(1) Let X be a noetherian scheme. Then $\mathsf{Spc}\,\mathsf{D}^{\mathsf{perf}}(R)$ is homeomorphic to X. In particular, radical thick tensor ideals of $\mathsf{D}^{\mathsf{perf}}(R)$ are bijectively corresponds to specialization-closed subsets of X.
(2) Let k be a field and G a finite group. Then $\mathsf{Spc}(\underline{\mathsf{mod}}\, kG)$ is homeomorphic to $\mathsf{Proj}\,\mathsf{H}^*(G;k)$. Here $\mathsf{H}^*(G;k)$ is the cohomology ring of G with coefficients in k. In paticular, radical thick tensor ideals of $\underline{\mathsf{mod}}\, kG$ are bijectively corresponds to specialization-closed subsets of $\mathsf{Proj}\,\mathsf{H}^*(G;k)$.

However, in these examples, the structure of Balmer spectra are determined by using "complete" classification of thick tensor ideals [3, 4, 6, 9, 10]. Without classification, the structure of Balmer spectra is difficult and mysterious. For instance, the complete classification of radical thick tensor ideals of $\mathsf{D}^-(\mathsf{mod}R)$ is not known and the structure of its Balmer spectrum is very difficult even in the case of DVR.

Example 8. [8, Theorem 7.11] Let $R = k[[x]]$ be a DVR. For any positive integer n, set

$$\mathcal{P}_n := \{M \in \mathsf{D}^-(\mathsf{mod}R) \mid ll_R(\mathsf{H}^{-i}(M)) \leq ci^{n-1} \text{ for some } c \geq 0 \text{ and any integer } i\},$$

where $ll_R(\mathsf{H}^i(M))$ denotes the Lowey length of $\mathsf{H}^i(M)$. Then

(1) $\mathcal{P}_n$ is a prime thick tensor ideal of $\mathsf{D}^-(\mathsf{mod}R)$,

(2) $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots$.

In particular, $\mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R)$ has infinite Krull dimension.

The motivation of this article is investigation of the structure of the Balmer spectrum of $\mathsf{D}^-(\mathsf{mod}R)$. As complete classification gives complete information on the Balmer spectrum, it is naturally expect that we obtain some information of the Balmer spectrum using "partial" classification. We use the following result in this direction.

Theorem 9. [8, Corollary 2.11 and Theorem 2.12]

(1) *There is a bijection between*
   (a) *the set of compactly generated thick tensor ideals of* $\mathsf{D}^-(\mathsf{mod}R)$*, and*
   (b) *the set of specialization-closed subsets of* $\mathsf{Spec}\,R$*.*
   Here, a compactly generated thick tensor ideal is the smallest thick tensor ideal containing some set of bounded complexes.

(2) *For any thick tensor ideal* $\mathcal{X}$*, there is a unique compactly generated thick tensor ideal* $\mathcal{C}$ *such that* $\mathsf{Supp}\,\mathcal{X} = \mathsf{Supp}\,\mathcal{C}$

Remark 10. Original motivation of this study is to investigate $\mathsf{D}^\mathsf{b}(\mathsf{mod}R)$. However, this problem is difficult since the category does not have any natural tensor structure. Thus, instead of considering the category, we enlarge to $\mathsf{D}^-(\mathsf{mod}R)$ and use its tensor structure.

3. Main Theorem

The main theorem of this article is the following result.

Theorem 11. *Let* $C \in \mathsf{D}^\mathsf{b}(\mathsf{mod}R)$ *be a bounded complex.*

(1) *There is a bijection between*
   (a) *the set of connected components of* $\mathsf{BSupp}\,C$*, and*
   (b) *the set of connected components of* $\mathsf{Supp}\,C$*.*

(2) *There is a bijection between*
   (a) *the set of irreducible components of* $\mathsf{BSupp}\,C$*, and*
   (b) *the set of irreducible components of* $\mathsf{Supp}\,C$*.*

Connectedness of $\mathsf{Supp}\,C$ of bounded complex C is characterized by indecomposability of C.

Lemma 12. *Let* $C \in \mathsf{D}^\mathsf{b}(\mathsf{mod}R)$ *be a bounded complex. If* C *is indecomposable, then the cohomological support* $\mathsf{Supp}\,C$ *is connected.*

Combining the main theorem and this lemma, we obtain the following result.

Corollary 13. *Let* $C \in \mathsf{D}^\mathsf{b}(\mathsf{mod}R)$ *be a bounded complex.*

(1) *If C is indecomposable, then* $\mathsf{BSupp}\, C$ *is connected. In particular, if R is indecomposable, then* $\mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R)$ *is connected.*
(2) *If* $\mathsf{Supp}\, C$ *is irreducible i.e., contains only one minimal element, then* $\mathsf{BSupp}\, C$ *is irreducible. In particular, if* $\mathsf{Spec}\, R$ *is irrecible, then* $\mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R)$ *is irreducible.*

Remark 14. Carlson [5] proved the corresponding result to this corollary for the stable category of a group ring and Balmer [2] generalized it to *rigid* tensor triangulated categories. We have to note that our category $\mathsf{D}^-(\mathsf{mod}R)$ is *never* rigid and hence our result is not included in Balmer's general result.

4. Sketch of the Proof

In this section, we will give a sketch of the proof of the main theorem. Let us begin with recalling some notions from point-set topology.

Definition 15. Let X be a topological space.

(1) We say that a subspace of X is a *clopen subset* if it is closed and open in X.
(2) A subspace W of X is said to be *specialization-closed* if for any $x \in W$, $\overline{\{x\}} \subseteq W$ holds.
(3) A subspace W of X is said to be *generalization-closed* if for any $x \in W$ and $y \in X$, $x \in \overline{\{y\}}$ implies $y \in W$.

Remark 16. (1) Every connected component is clopen.
(2) Let W be a subspace of $\mathsf{Spec}\, R$. Then W is specialization closed (resp. generalization closed) in $\mathsf{Spec}\, R$ if and only if

$$\mathfrak{p} \in W,\ \mathfrak{p} \subseteq \mathfrak{q} \Longrightarrow \mathfrak{q} \in W \ \ (\text{resp. } \mathfrak{q} \in W,\ \mathfrak{p} \subseteq \mathfrak{q} \Longrightarrow \mathfrak{p} \in W).$$

(3) [1, Proposition 2.9] Let $\mathcal{T}$ be an essentially small tensor triangulated category and W a subspace of $\mathsf{Spc}\,\mathcal{T}$. Then W is specialization closed (resp. generalization closed) in $\mathsf{Spc}\,\mathcal{T}$ if and only if

$$\mathcal{P} \in W,\ \mathcal{P} \supseteq \mathcal{Q} \Longrightarrow \mathcal{Q} \in W \ \ (\text{resp. } \mathcal{Q} \in W,\ \mathcal{P} \supseteq \mathcal{Q} \Longrightarrow \mathcal{P} \in W).$$

In [8], authors introduced two maps between the Balmer spectrum and the Zariski spectrum:

$$\mathfrak{s} : \mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R) \rightleftarrows \mathsf{Spec}\, R : \mathcal{S}.$$

Here, the map $\mathcal{S}$ is defined in Example 4 and $\mathfrak{s}(\mathcal{P})$ is a unique maximal element of the set of ideal I with $R/I \in \mathcal{P}$, see [8, Proposition 3.7]. Using these map, we will compare the Balmer spectrum and the Zariski spectrum. For this, let me list properties of these maps.

Proposition 17. [8, Theorem 3.9, Corollary 3.10, Theorem 4.5]

(1) $\mathfrak{s}$ *and* $\mathcal{S}$ *are order-reversing maps with respect to the inclusion relation.*
(2) $\mathfrak{s}$ *is continuous.*
(3) $\mathfrak{s} \cdot \mathcal{S} = 1$. *In particular,* $\mathfrak{s}$ *is surjective and* $\mathcal{S}$ *is injective.*
(4) *For any* $\mathcal{P} \in \mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R)$,

$$\mathsf{Supp}\,\mathcal{P} = \{\mathfrak{p} \in \mathsf{Spec}\, R \mid \mathfrak{p} \not\subseteq \mathfrak{s}(\mathcal{P})\}.$$

(5) *For any* $\mathcal{P} \in \mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R)$, *one has*

$$\mathcal{S}(\mathfrak{s}(\mathcal{P})) = \mathsf{Supp}^{-1}(\mathsf{Supp}\,\mathcal{P}) \supseteq \mathcal{P}.$$

Let C be a bounded complex. By Theorem 9, $C \in \mathcal{X}$ if and only if $\mathsf{Supp}\,C = \mathsf{Supp}\,\mathcal{X}$ for any thick tensor ideal $\mathcal{X}$. Therefore, by Proposition 17(5), $\mathcal{P} \in \mathsf{BSupp}\,C$ if and only if $\mathcal{S}(\mathfrak{s}(\mathcal{P})) \in \mathsf{BSupp}\,C$. Using these observation, we can check that

$$\mathfrak{s} : \mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R) \rightleftarrows \mathsf{Spec}\,R : \mathcal{S}$$

restricts to

$$\mathfrak{s} : \mathsf{BSupp}\,C \rightleftarrows \mathsf{Supp}\,C : \mathcal{S}.$$

The following two lemmata are key to prove the main theorem.

Lemma 18. *The above pair of maps induce a one-to-one correspondence*

$$\mathfrak{s} : \mathsf{Max}\,\mathsf{BSupp}\,C \rightleftarrows \mathsf{Min}\,\mathsf{Supp}\,C : \mathcal{S}.$$

Here, $\mathsf{Max}\,\mathsf{BSupp}\,C$ *(resp.* $\mathsf{Min}\,\mathsf{Supp}\,C$*) is the set of maximal (resp. minimal) elements of* $\mathsf{BSupp}\,C$ *(resp.* $\mathsf{Supp}\,C$*).*

Proof. Because $\mathcal{S} : \mathsf{Spec}\,R \to \mathsf{Spc}\,\mathsf{D}^-(\mathsf{mod}R)$ is injective, we have only to check that the map $\mathcal{S} : \mathsf{Min}\,\mathsf{Supp}\,C \to \mathsf{Max}\,\mathsf{BSupp}\,C$ is well-defined and surjective. Let $\mathfrak{p}$ be a minimal element of $\mathsf{Supp}\,C$. We show that $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\,C$. Take a prime thick tensor ideal $\mathcal{P}$ in $\mathsf{BSupp}\,C$ containing $\mathcal{S}(\mathfrak{p})$. Then $\mathfrak{s}(\mathcal{P}) \subseteq \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{p}$ by Proposition 17. Since both $\mathfrak{p}$ and $\mathfrak{s}(\mathcal{P})$ belong to $\mathsf{Supp}\,C$ by the above argument, the minimality of $\mathfrak{p}$ shows the equality $\mathfrak{p} = \mathfrak{s}(\mathcal{P})$. Hence, we have

$$\mathsf{Supp}\,\mathcal{P} = \{\mathfrak{q} \in \mathsf{Spec}\,R \mid \mathfrak{q} \not\subseteq \mathfrak{s}(\mathcal{P}) = \mathfrak{p} = \mathfrak{s}(\mathcal{S}(\mathfrak{p}))\} = \mathsf{Supp}\,C\mathcal{S}(\mathfrak{p}).$$

This shows that $\mathcal{P} \subseteq \mathcal{S}(\mathfrak{p})$ and thus $\mathcal{S}(\mathfrak{p})$ is a maximal element in $\mathsf{BSupp}\,C$. For this reason, the map $\mathcal{S} : \mathsf{Min}\,\mathsf{Supp}\,C \to \mathsf{Max}\,\mathsf{BSupp}\,C$ is well-defined.

Next we check the surjectivity of the map $\mathcal{S} : \mathsf{Min}\,\mathsf{Supp}\,C \to \mathsf{Max}\,\mathsf{BSupp}\,C$. Let $\mathcal{P}$ be a maximal element of $\mathsf{BSupp}\,C$. It follows from the argument before this lemma that $\mathcal{S}\mathfrak{s}(\mathcal{P})$ is also an element in $\mathsf{BSupp}\,C$. On the other hand, $\mathcal{S}\mathfrak{s}(\mathcal{P})$ contains $\mathcal{P}$ by Proposition 17(3). Thus, we get $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P})$ from the maximality of $\mathcal{P}$. Let $\mathfrak{p}$ be an element of $\mathsf{Supp}\,C$ with $\mathfrak{p} \subseteq \mathfrak{s}(\mathcal{P})$. Then $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P}) \subseteq \mathcal{S}(\mathfrak{p})$. Since $\mathcal{P}$ is maximal in $\mathsf{BSupp}\,C$, one has $\mathcal{P} = \mathcal{S}(\mathfrak{p})$. Hence, $\mathfrak{p} = \mathfrak{s}\mathcal{S}(\mathfrak{p}) = \mathfrak{s}(\mathcal{P})$ and this shows that $\mathfrak{s}(\mathcal{P})$ is a minimal element of $\mathsf{Supp}\,C$. As a result, one has $\mathcal{S}(\mathfrak{p}) = \mathcal{S}\mathfrak{s}(\mathcal{P}) = \mathcal{P}$ and this shows that $\mathcal{S} : \mathsf{Min}\,\mathsf{Supp}\,C \to \mathsf{Max}\,\mathsf{BSupp}\,CC$ is surjective. $\square$

Lemma 19. *Let* X *is either* $\mathsf{BSupp}\,C$ *or* $\mathsf{Supp}\,C$*. Then subset* W *of* X *is closed and open if and only if* W *is specialization-closed and generalization-closed.*

Proof. We show this statement only for $X = \mathsf{BSupp}\,C$ because a similar argument works also for $X = \mathsf{Supp}\,C$. By symmetry, we need to check that W is closed.

The key point is to prove the following equation

$$W = \bigcup_{\mathcal{P} \in \mathsf{Max}\,\mathsf{BSupp}\,C \cap W} \overline{\{\mathcal{P}\}}$$

Since W is specialization closed, $W \supseteq \bigcup_{\mathcal{P} \in \mathsf{Max}\,\mathsf{BSupp}\,C \cap W} \overline{\{\mathcal{P}\}}$ holds. Let $\mathcal{P}$ be an element of W. Take a minimal element $\mathfrak{p}$ in $\mathsf{Supp}\,C$ contained in $\mathfrak{s}(\mathcal{P})$. We can take such a $\mathfrak{p}$ since $\mathsf{Supp}\,C$ is a closed subset of $\mathsf{Spec}\,R$. Then

$$\mathcal{P} \subseteq \mathcal{S}\mathfrak{s}(\mathcal{P}) \subseteq \mathcal{S}(\mathfrak{p}).$$

By Lemma 18, $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\,C$. Moreover, $\mathcal{S}(\mathfrak{p})$ belongs to W since W is generalization closed and $\mathcal{P} \in W$. These show that $\mathcal{S}(\mathfrak{p})$ is a maximal element of $\mathsf{BSupp}\,C$. Accordingly, we obtain $\mathcal{P} \in \overline{\{\mathcal{S}(\mathfrak{p})\}}$ with $\mathcal{S}(\mathfrak{p}) \in \mathsf{Max}\,\mathsf{BSupp}\,C$ and hence the converse inclusion holds true.

Note that $\mathsf{BSupp}\,C$ is closed and thus contains only finitely many minimal elements. By using the one-to-one correspondence in Lemma 18, $\mathsf{Max}\,\mathsf{BSupp}\,C$ is also a finite set. Consequently, W is a finite union of closed subsets, and hence is closed. □

Using this lemma, the topological problem is interpreted to combinatrial problem. We can prove the following two lemmas using this technique.

Lemma 20. *Let U be a clopen subset of $\mathsf{BSupp}\,C$. Then*

(1) $\mathfrak{p} \in \mathfrak{s}(U)$ *if and only if* $\mathcal{S}(\mathfrak{p}) \in U$.

(2) $\mathfrak{s}(U)$ *is a clopen subset in* $\mathsf{Supp}\,C$.

Lemma 21. *Let U be a clopen subset of $\mathsf{BSupp}\,C$. Then $\mathfrak{s}^{-1}\mathfrak{s}(U) = U$.*

Now, we are ready to prove the main theorem.

Proof of the Main Theorem. (1) By Lemma 20(2), we obtain a well-defined order-preserving map

$$\{\text{clopen subsets of } \mathsf{BSupp}\,C\} \to \{\text{clopen subsets of } \mathsf{Supp}\,C\}, U \mapsto \mathfrak{s}(U).$$

This map is injective by Lemma 21 and surjective since $\mathfrak{s}$ is continuous and sujective.

Note that our topological spaces $\mathsf{BSupp}\,C$ and $\mathsf{Supp}\,C$ have only finitely many connected components by Lemma 18 and the proof of Lemma 20. Thus, connected components are nothing but minimal non-empty clopen subsets. Therefore, the above order-preserving bijection restricts to the bijection what we want.

(2) By [1, Proposition 2.9], irreducible components of $\mathsf{BSupp}\,C$ are bijectively correspond to its maximal elements. Thus, the statement follows from Lemma 18. □

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